THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4230 2024-25
Lecture 6
January 22, 2025 (Wednesday)

1 Recall

In the previous lectures, we discuss the followings:

Mangasarian Fromovitz Qualification condition:

① the family of vectors $\{\nabla h_1(x), \ldots, \nabla h_m(x)\}$ is linearly independent.

② there exists a vector $v \in \mathbb{R}^n$ satisfying

$$\langle \nabla h_j(x^*), v \rangle = 0, \ \forall j = 1, \ldots, m$$

and

$$\langle \nabla g_i(x^*), v \rangle < 0, \ \forall i \in I(x) := \{k : g_k(x) = 0\}.$$

Then the constraint K is qualified at $x \in K$.

Abadie's Condition:

$$T_K(x) = \left\{ v \in \mathbb{R}^n : \ \exists (s_k, v_k) \to (0^+, v) \ \text{ and } \ x + s_k v_k \in K \right\}$$
$$D = \left\{ v \in \mathbb{R}^n : \begin{array}{c} \langle \nabla g_i(x), v \rangle \leq 0, \ \forall i = 1, \ldots, \ell \ \text{ satisfying } \ g_i(x) = 0 \\ \langle \nabla h_j(x), v \rangle = 0, \ \ \forall j = 1, \ldots, m \end{array} \right\}$$

If $T_K(x) = D$, then the constraint K is qualified at $x \in K$.

The problem (P) and the *feasible* set K as follows:

$$\inf_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{cases} g_i(x) \leq 0, & i = 1, \ldots, \ell \\ h_j(x) = 0, & j = 1, \ldots, m \end{cases} \tag{P}$$

where $f, g_i, h_j \in C^1$, and

$$K = \{x \in \mathbb{R}^n : g_i(x) \leq 0, \ h_j(x) = 0, \ i = 1, \ldots, \ell, \ j = 1, \ldots, m\}$$

Also, together with optimal solution $\boldsymbol{x^*}$ and the **qualification**, we have the following:

Let $x^* \in K$ be a solution to (P) and assume that K is **qualified** at x^*. Then there exists $\lambda_1, \cdots, \lambda_\ell \geq 0$ and $\mu_1, \cdots, \mu_m \in \mathbb{R}$ such that

$$\begin{cases} \displaystyle\sum_{i=1}^{\ell} \lambda_i g_i(x^*) = 0 \\ \displaystyle\nabla f(x^*) + \sum_{i=1}^{\ell} \lambda_i \nabla g_i(x^*) + \sum_{j=1}^{m} \mu_j \nabla h_j(x^*) = \mathbf{0} \end{cases}$$

2 Exercise

Example 1. Consider a *feasible set* $K = \{(x,y) \in \mathbb{R}^2 : \ y \geq x^2, \ y \leq 0\} = \{(0,0)\}$. Discuss the qualification of the set K at $x^* \in K$.

Solution. Let $g_1(x,y) = x^2 - y$ and $g_2(x,y) = y$. Then, we have

$$\nabla g_1(x,y)\big|_{(0,0)} = \begin{pmatrix} 2x \\ -1 \end{pmatrix}_{(0,0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\nabla g_2(x,y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$T_K(0) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Also, we have

$$\begin{aligned} \left\{ v \in \mathbb{R}^2 : \langle \nabla g_i(x,y), v \rangle \leq 0 \right\} &= \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : \begin{matrix} v_1 - v_2 \leq 0 \\ v_1 + v_2 \leq 0 \end{matrix} \right\} \\ &= \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : v_1 \leq 0, \ v_2 \in [v_1, -v_1] \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \end{aligned}$$

By the Abadie's Condition, K is qualified at $x^* = (0,0) \in K$. ◀

Example 2. Discuss the qualification of

$$K = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : y \leq x^3, \ \ y \geq 0 \right\}$$

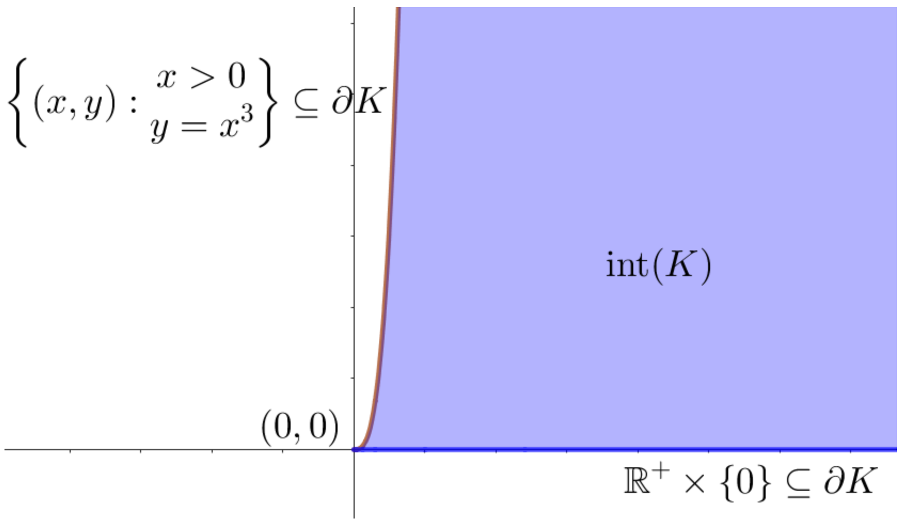

Figure 1: Example 2

Solution. Let $g_1(x,y) = y - x^3$ and $g_2(x,y) = -y$.
Now, we consider the following cases for points in K:

- **Case 1:** For any $(x,y) \in \text{int}(K)$
  Then we have $T_K(x) = \mathbb{R}^2 = D$.

- **Case 2:** For $x > 0,\ y = 0$.
  Then $g_1(x,y) < 0,\ g_2(x,y) = 0$.
  The tangent cone becomes
  $$T_K(x) = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : \begin{matrix} v_1 \in \mathbb{R} \\ v_2 \geq 0 \end{matrix} \right\}$$
  and
  $$D = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : \left\langle \nabla g_2(x,y), \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle \leq 0 \right\} = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : -v_2 \leq 0 \right\}$$

- **Case 3:** For $x > 0,\ y = x^3$.
  The tangent cone of K at those points is given by
  $$T_K(x,y) = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : v_2 \leq 3x^2 \cdot v_1 \right\}$$
  and
  $$D = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : \left\langle \begin{pmatrix} -3x^2 \\ 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle \leq 0 \right\} = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : v_2 - 3x^2 \cdot v_1 \leq 0 \right\} = T_K(x,y)$$

- **Case 4:** At $(x,y) = (0,0)$. Then
  $$T_K(x,y) = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : v_1 \geq 0,\ v_2 = 0 \right\}$$
  and
  $$D = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : \begin{matrix} \left\langle \begin{pmatrix} -3x^2 \\ 1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle \leq 0 \\ \left\langle \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle \leq 0 \end{matrix} \right\} = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : v_2 \leq 3x^2 v_1,\ v_2 \geq 0 \right\}$$
  $$= \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : v_1 \in \mathbb{R},\ v_2 = 0 \right\}$$

Thus, K is qualified in both cases 1,2,3 and not qualified at $(0,0)$ in case 4. ◀

Example 3. Discuss the qualification on the set
$$K = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \mathbb{R}^2 :\ y \geq x^2,\ y \leq 0 \right\}$$

Solution. Let $g_1(x,y) = x^2 - y$ and $g_2(x,y) = y$.
Also, it is clear that
$$K = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \left\{ (x,y) : x^2 + y^2 = 0 \right\}$$

Remarks. 1. For the same set, there are different ways to formulate the constraints, the set D, it depends on the chosen constraints.

2. However, $T_K(x)$ is geometric, it does not depend on the constraints chosen!

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— End of Lecture 6 —